3/16/26 Quiz Wednesday

Start §3.3 . the dimension of a subspace of $\mathbb{R}^n$ (Important stuff in this section.)

Last time we stated:

Theorem (3.3.1)

Let V be a subspace of $\mathbb{R}^n$. Let $\vec{v}_1, \dots, \vec{v}_p \in V$ be lin. indep and $\vec{w}_1, \dots, \vec{w}_q$ span V. Then $p \leq q$

Let's prove it.

No surprise: Intuition: as you add vectors, you can only stay lin indep until they span, but once they span you can keep adding.

proof kind of different proof (cute)

Let $A = \underbrace{\left[\vec{v}_1 \middle| \dots \middle| \vec{v}_p \right]}_{n \text{ rows, } p \text{ cols}}$ $\boxed{(n \times p)}$ and $B = \underbrace{\left[\vec{w}_1 \middle| \dots \middle| \vec{w}_q \right]}_{n \text{ rows, } q \text{ cols}}$ $\boxed{n \times q}$ (both have n rows)

We know

- $\ker(A) = \vec{0}$ since the $\vec{v}_i$ are lin indep
- the col. space of B is V since the $\vec{w}_i$ span V.

Since $\{\vec{w}_1, \dots, \vec{w}_q\}$ span V and each $\vec{v}_i$ is in V, we can write

$$\vec{v}_1 = a_{11}\vec{w}_1 + \dots + a_{q1}\vec{w}_q$$
$$\vec{v}_2 = a_{12}\vec{w}_1 + \dots + a_{q2}\vec{w}_q$$
$$\vdots$$
$$\vec{v}_p = a_{1p}\vec{w}_1 + \dots + a_{qp}\vec{w}_q$$

Rewrite these as matrix products

1st one:

$$\underbrace{\begin{bmatrix} \vec{w}_1 & | & \cdots & | & \vec{w}_q \end{bmatrix}}_{B} \begin{bmatrix} a_{11} \\ \vdots \\ a_{q1} \end{bmatrix} = \vec{v}_1$$

2nd one:

$$\underbrace{\begin{bmatrix} \vec{w}_1 & | & \cdots & | & \vec{w}_q \end{bmatrix}}_{B} \begin{bmatrix} a_{12} \\ \vdots \\ a_{q2} \end{bmatrix} = \vec{v}_2$$

$$\vdots$$

pth one

$$\underbrace{\begin{bmatrix} \vec{w}_1 & | & \cdots & | & \vec{w}_q \end{bmatrix}}_{B} \begin{bmatrix} a_{1p} \\ \vdots \\ a_{qp} \end{bmatrix} = \vec{v}_p$$

Let $\vec{u}_1 = \begin{bmatrix} a_{11} \\ \vdots \\ a_{q1} \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} a_{12} \\ \vdots \\ a_{q2} \end{bmatrix}$, $\dots$, $\vec{u}_p = \begin{bmatrix} a_{1p} \\ \vdots \\ a_{qp} \end{bmatrix} \in \mathbb{R}^q$

Conclusion so far:

$\vec{v}_1 = B\vec{u}_1$

$\vec{v}_2 = B\vec{u}_2$

$\vdots$

$\vec{v}_p = B\vec{u}_p$

Put this together:

$$A = \left[\vec{v}_1 \,\middle|\, \cdots \,\middle|\, \vec{v}_p\right] = B\underbrace{\left[\vec{u}_1 \,\middle|\, \cdots \,\middle|\, \vec{u}_p\right]}_{\text{call this } C}$$

reality check: $n \times p$ $\quad (n \times q) \quad (q \times p)$

So $\boxed{A = BC}$

Now a couple of observations

1. $\ker(C) \subset \ker(A)$ since if $\vec{v} \in \ker(C)$ then

$$C\vec{v} = \vec{0} \Rightarrow B(C\vec{v}) = B\vec{0} = \vec{0}$$
$$\Rightarrow A\vec{v} = 0 \quad \text{(associativity)}$$

2. $\ker(A) = \{\vec{0}\}$ since the $\vec{v}_i$ are lin. indep (we already observed this)

3. $\therefore \ker(C) = \{\vec{0}\}$

But we showed on 2/27:

<u>theorem</u> Suppose A is an $n \times m$ matrix with $n \neq m$.

(a) if $\operatorname{rank}(A) = m$ then $\ker(A) = \{\vec{0}\}$

(b) Conversely, if $\ker(A) = \{\vec{0}\}$ then $\operatorname{rank}(A) = m$

(c) if $\ker(A) = \{\vec{0}\}$ then $\underset{\#\text{cols}}{m} \leq \underset{\#\text{rows}}{n}$

we didn't need the assumption $n \neq m$. It's just that we did the square case previously. we can ignore it.

So here apply it to C, instead of A.

C is a $q \times p$ matrix and $\ker(C) = \{\vec{0}\}$

$\Rightarrow p \leq q$ //

<u>Corollary</u> (important!) (Thm 3.3.2)

Let V be a vector subspace of $\mathbb{R}^n$. Then all bases of V have the same number of elements!

<u>proof</u>

If $\vec{v}_1, \ldots, \vec{v}_p$ and $\vec{w}_1, \ldots, \vec{w}_q$ are both bases, apply the theorem twice, interchanging roles. So $p \leq q$ <u>and</u> $q \leq p$. //

<u>Def</u> If V is a vector subspace of $\mathbb{R}^n$ then the <u>dimension</u> of V is the # of elements of a basis. write $\dim(V)$.

Note you don't have to be able to visualize it to understand it!

Ex A plane in $\mathbb{R}^3$ has dim 2. Easier to see using the following:

Theorem (3.3.4)

Let V be a subspace of $\mathbb{R}^n$ with $\boxed{\dim(V) = m}$. Then

(a) We can find at most m lin. indep vectors in V

(b) We need at least m vectors to span V.

(c) If $\boxed{m}$ vectors in V are lin indep then they span V.

(d) If $\boxed{m}$ vectors in V span V then they're lin indep

so if you have the right # of vtrs and know one of the 2 properties, the other follows automatically

proof of (a) + (c) - parts (b) and (d) are in HW

(a) Say $\vec{v}_1, \ldots, \vec{v}_p$ are lin indep in V. Let $\vec{w}_1, \ldots, \vec{w}_m$ be a basis of V.

By last Theorem, $p \le m$.

prove (c) next time